Name: ______________________________

Teacher: ____________________________

Class: ______________________________

FORT STREET HIGH SCHOOL

2010

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 2

Mathematics Extension 1

TIME ALLOWED: 1½ HOURS

Outcomes Assessed	Questions	Marks
Manipulates algebraic expressions to solve problems involving inverse functions	5, 6	
Synthesises mathematical solutions to harder problems and communicates them in appropriate form	1, 2, 3, 4	

Question	1	2	3	4	5	6	Total	%
Marks	/9	/11	/8	/12	/10	/10	/60	

Directions to candidates:

- Attempt all questions
- The marks allocated for each question are indicated
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used
- **Each new question is to be started in a new booklet**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (9 marks) Use a SEPARATE writing booklet

a) Prove the identity $\dfrac{1-tan^2 A}{1+tan^2 A}=\cos 2A$. [2]

b) The point $P\left(-3,8\right)$ divides the interval AB externally in the ratio $k:1$. If A is the point $(6,-4)$ and B is the point $(0,4)$, find the value of k. [2]

c) Solve for x: $\left|\dfrac{5}{x}+3\right|>3$. [3]

d) When the polynomial function $f(x)$ is divided by x^2-16, the remainder is $3x-1$. What is the remainder when $f(x)$ is divided by $x-4$? [2]

Question 2 (11 marks) Use a SEPARATE writing booklet

a) Find the exact value of $\sin 105°$. [2]

b) Find all values of θ for which $\tan\theta = 3\cot\theta$. (give your answer in radians). [2]

c) Show that $\cos(A+B)+\cos\left(A-B\right)=2\cos A\cos B$. Hence or otherwise, evaluate

$$\int_0^{\frac{\pi}{6}} 2\cos 3x\cos 2x \ \ dx.$$

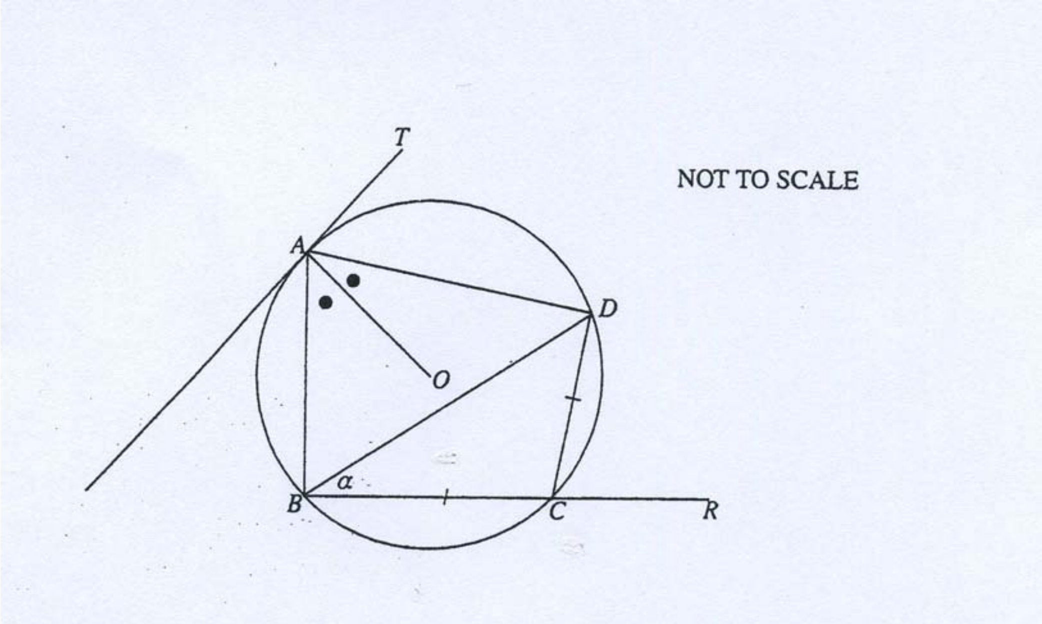
[3]

d) Express $\sin x$ and $\cos x$ in terms of $t=\tan\dfrac{x}{2}$ and hence find the solution of $5\cos x-2\sin x=2$ where $0\le x\le 2\pi$. [4]

Question 3 (8 marks) Use a SEPARATE writing booklet

a) Evaluate $\lim_{\theta \to 0} \dfrac{\theta + \sin 2\theta}{3\theta}$ [3]

b)



Points A, B, C and D lie on a circle centre O. The line TA is a tangent to the circle at A, and BC is produced to R. The interval OA bisects $\angle$ BAD, and BC = CD. The size of $\angle$ DBC is α .

Copy or trace the diagram into your writing booklet.

i) Explain why $\angle$ DCR = 2α . [1]

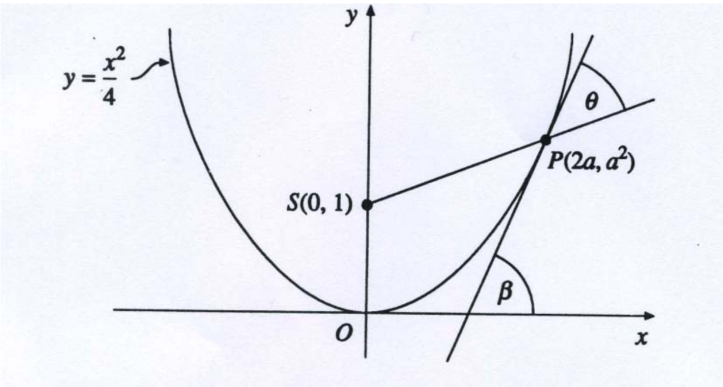
ii) Show that $\angle$ OAD = α . [2]

iii) Prove that $\angle$ ABC is a right angle. [2]

Question 4 (12 marks) Use a SEPARATE writing booklet

a) For $P(x) = x^3 - 6x^2 + ax - 4$ and $a > 0$ given that all the roots of $P(x) = 0$ are real and positive, and that one of the roots is the product of the other two, show $a = 10$. [4]

b)



Let $P(2a, a^2)$ be a point on the parabola

$$y = \frac{x^2}{4},$$

and let S be the point (0, 1). The tangent to the parabola at P makes an angle of β with the x axis. The angle between SP and the tangent is θ. Assume that $a > 0$, as indicated.

i) Show that $\tan\beta = a$. [1]

ii) Show that the gradient of SP is $\frac{1}{2}\left(a - \frac{1}{a}\right)$. [1]

iii) Show that $\tan\theta = \frac{1}{a}$. [2]

iv) Hence find the value of $\theta + \beta$. [2]

v) Find the coordinates of P if $\theta = \beta$. [2]

Question 5 (10 marks) Use a SEPARATE writing booklet

a) Sketch $y = 4\sin^{-1}(3x)$ clearly showing the domain and range and any intercepts. [2]

b) Consider the function $f(x) = x(x-2)^2$.

(i) Sketch the function $f(x)$ given that stationary points exist at $x = \frac{2}{3}$ and $x = 2$. [2]

(ii) Now consider $f(x) = x(x-2)^2$, where $x \le a$ (a is a constant). Find the value of a for which $f^{-1}(x)$ can exist. Give a reason for your answer. [2]

(iii) State the domain of $f^{-1}(x)$. [1]

c) Find the exact values of x and y which satisfy the simultaneous equations

$\sin^{-1} x - \cos^{-1} y = \frac{\pi}{12}$ and $\sin^{-1} y + \cos^{-1} x = \frac{5\pi}{12}$ [3]

Question 6 (10 marks) Use a SEPARATE writing booklet

(a) Find the primitive function of $\dfrac{1}{\sqrt{4-x^2}}$ [1]

(b) Differentiate $\cos^{-1}(3-2x)$, giving your answer in the simplest factored form. [3]

(c) Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+9x^2}$ giving your answer in exact form. [2]

(d) The region in the first quadrant bounded by the curve $y = 3\tan^{-1} x$ and the y-axis between y = 0 and y = π is rotated through one complete revolution about the y-axis. Find the volume of the solid of revolution in exact form. [4]

SOLUTIONS EXT. 1 ASSESSMENT TASK 2 2010

Name: Question 1 /9 Teacher:

a) $\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A} \div \dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}$ ✓

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A}$ ✓

$= \cos^2 A - \sin^2 A$

$= \cos 2A$

OR

$\dfrac{1-\tan^2 A}{1+\tan^2 A}$

$= \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A} \times \dfrac{1}{\sec^2 A}$ ✓

$= \dfrac{\cos^2 A - \sin^2 A}{\cancel{\cos^2 A}} \times \cancel{\cos^2 A}$

$= \cos^2 A - \sin^2 A$ ✓

$= \cos 2A$.

Comments

1 for writing expression in terms of $\cos A$, $\sin A$

1 for knowing this is equal to $\cos 2A$.

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Name: Question 1 ct'd. Teacher:

b) $P(-3, 8)$ $A(6, -4)$ $B(0, 4)$

$k : -1$ (external division)

$$x = \frac{kx_2 + lx_1}{k+l}$$

$$-3 = \frac{k(0) - 1(6)}{k-1} \checkmark$$

$-3k + 3 = -6$

$-3k = -9$

$k = 3$ ✓

1 for substituting correctly

1 for value for k.

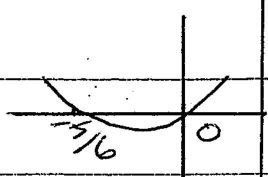
c) $\left| \frac{5}{x} + 3 \right| > 3$

1 for each case & solution (2 marks)

1 for overall solution

note: $x \neq 0$

$\frac{5}{x} + 3 > 3$ or	$\frac{5}{x} + 3 < -3$
$\frac{5}{x} > 0$	$\frac{5}{x} < -6$
$5x > 0$	$5x < -6x^2$
$x > 0$ ✓	$6x^2 + 5x < 0$
	$x(6x+5) < 0$
	$-5/6 < x < 0$ ✓



∴ overall solution is $x > -5/6$, $x \neq 0$ ✓

OR $\left|\dfrac{5}{x}+3\right| > 3$

note: $x \neq 0$

when $\dfrac{5}{x}+3>3$ or $\dfrac{5}{x}+3<-3$

$\dfrac{5}{x}>0$ $\qquad$ $\dfrac{5}{x}<-6$

$\therefore$ $x>0$ ✓ $\qquad$ ~~If~~ $x<0$,

$5>-6x$

$-\dfrac{5}{6}<x$

i.e. $-5/6<x<0$ ✓

$\therefore$ overall solution is $x>-5/6,\ x\neq 0$ ✓

d) $f(x)=(x^2-16)Q(x)+(3x-1)$ ✓

$f(4)=(4^2-16)Q(4)+(12-1)$

$= \quad 0 \ +11$

$=11$

$\therefore$ remainder is 11. ✓

1 mark for expressing $f(x)$ in terms of (divisor)(quotient) + remainder

1 mark for remainder.

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Name: Question 2 /11 Teacher:

a) $\sin 105° = \sin(60°+45°)$

$= \sin 60° \cos 45° + \cos 60° \sin 45°$ ✓ 1 mark for expansion

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2} \times \frac{1}{\sqrt{2}}\right)$ 1 mark for exact value

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$ ✓

OR

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ✓

b) $\tan\theta = 3\cot\theta$ 1 mark for obtaining $\tan\theta = \pm\sqrt{3}$

$\tan^2\theta = 3$ 1 mark for both general solutions

$\tan\theta = \sqrt{3}$, $\tan\theta = -\sqrt{3}$ ✓

$\therefore \theta = n\pi + \frac{\pi}{3}$ or $\theta = n\pi + \frac{2\pi}{3}$ ✓

c) $\cos(A+B) + \cos(A-B)$

$= \cos A\cos B - \cancel{\sin A \sin B} + \cos A \cos B + \cancel{\sin A \sin B}$ 1 mark for expanding

$= 2\cos A \cos B$ ✓

$\therefore \int_0^{\pi/6} 2\cos 3x \cos 2x \, dx$ 1 mark for primitive function

$= \int_0^{\pi/6} (\cos 5x + \cos x)\, dx$ 1 mark for value

$= \left[\frac{1}{5}\sin 5x + \sin x\right]_0^{\pi/6}$ ✓

$= \left[\frac{1}{5}\sin\frac{5\pi}{6} + \sin\frac{\pi}{6}\right] - 0$

$= \frac{1}{5} \times \frac{1}{2} + \frac{1}{2} = \frac{3}{5}$ ✓

Name: ______ Teacher: ______

d) $\sin x = \dfrac{2t}{1+t^2}$ ✓

$\cos x = \dfrac{1-t^2}{1+t^2}$

1 mark for expressing $\sin x$ & $\cos x$ in terms of t

$5\cos x - 2\sin x = 2$

$5\left(\dfrac{1-t^2}{1+t^2}\right) - 2\left(\dfrac{2t}{1+t^2}\right) = 2$

$5 - 5t^2 - 4t = 2 + 2t^2$ ✓

$7t^2 + 4t - 3 = 0$

1 mark for attempting to simplify equation in terms of t to simple quadratic form

$(7t-3)(t+1) = 0$

$\therefore t = \dfrac{3}{7}$ or $t = -1$

$\tan\dfrac{x}{2} = \dfrac{3}{7}$, $\tan\dfrac{x}{2} = -1$ ✓

1 mark here

Since $0 \le x \le 2\pi$, $0 \le \dfrac{x}{2} \le \pi$

$\dfrac{x}{2} \doteqdot 0.4049$, $\dfrac{3\pi}{4}$

$\therefore x = 0.8098$, $\dfrac{3\pi}{2}$ ✓

1 mark for both correct solutions for x

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Name: Question 3 $\overline{8}$ Teacher:

a) $\lim_{\theta \to 0} \frac{\theta + \sin 2\theta}{3\theta}$

1 mark for deducing how to use fundamental limit to help evaluate

$= \lim_{\theta \to 0} \frac{1}{3} + \frac{2}{3}\frac{\sin\theta\cos\theta}{\theta}$ ✓

$= \frac{1}{3} + \frac{2}{3} \times 1 \times 1$ ✓

$= 1$ ✓

2 marks for working

OR

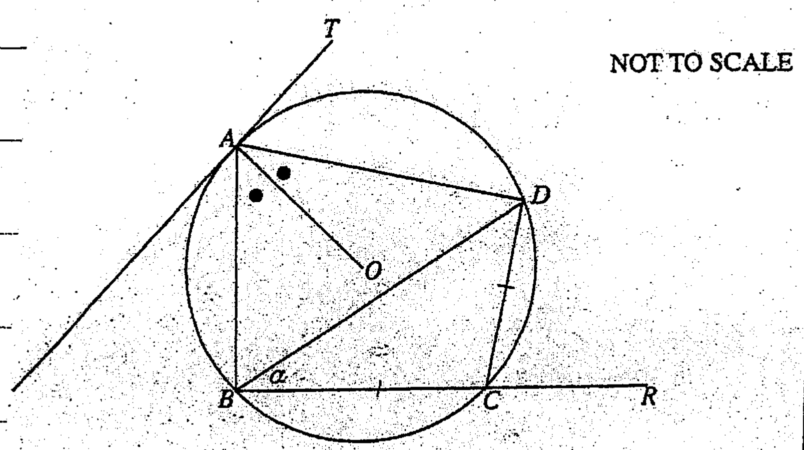
$\lim_{\theta \to 0} \frac{\theta + \sin 2\theta}{3\theta}$

$= \lim_{\theta \to 0} \frac{1}{3} + \frac{\sin 2\theta}{2\theta} \times \frac{2}{3}$ ✓

$= \frac{1}{3} + 1 \times \frac{2}{3}$ ✓

$= 1$ ✓

b)



i) 1 mark for saying ext. ∠ of ΔDCB)

ii) 1 mark for realising $\angle BAD = \angle DCR$ + 1 mark for reason for this

iii) marks shown over.

i) $\angle BDC = \alpha$ (base ∠s of isosceles ΔDCB) ✓

$\therefore \angle DCR = 2\alpha$ (exterior ∠ of ΔDCB)

b) ii) $\angle BAD = \angle DCR = 2\alpha$ ✓

(exterior $\angle$ of cyclic quad^{l} ABCD ✓ equals interior opposite $\angle$)

$\therefore \angle OAD = \alpha$ (given OA bisects $\angle BAD$)

iii) $\angle TAD = 90° - \alpha$

(radius $\perp$ tangent at pt. of contact) ✓ 1 mark

$\angle ABD = 90° - \alpha$

($\angle$ in alternate segment) 1 mark.

$\therefore \angle ABC = 90° - \alpha + \alpha = 90°$ ✓

a) $P(x) = x^3 - 6x^2 + ax - 4$, $a > 0$	
roots: $\alpha, \beta, \alpha\beta$ real & positive!	
$\alpha + \beta + \alpha\beta = 6$ (1)	1 mark.
$\alpha\beta + \alpha\beta^2 + \alpha^2\beta = \alpha\beta(1 + \beta + \alpha) = a$ (2)	
$(\alpha\beta)^2 = 4$ (3)	
$\therefore \alpha\beta = 2$ sub into (1)	1 mark for $\alpha\beta = 2$
$\therefore \alpha + \beta = 4$ sub into (2)	1 mark for $\alpha + \beta = 4$
$2(1 + 4) = a$	1 mark for finding 'a'.
$\therefore a = 10$.	
b) i) $y = \frac{x^2}{4}$	1 mark for saying tangent at P has $m = a$ & using $m = \tan\theta$.
$y' = \frac{x}{2}$ at $x = 2a$,	
$y' = a$ (m of tangent at P)	
Now $m = \tan\theta$	
$a = \tan\beta$.	
ii) $m_{SP} = \frac{a^2 - 1}{2a}$	1 mark for using gradient formula & rearranging to obtain answer given.
$= \frac{1}{2}\left(\frac{a^2}{a} - \frac{1}{a}\right)$	
$= \frac{1}{2}\left(a - \frac{1}{a}\right)$	

Name: Question 4 ct'd. Teacher:

iii) $\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$ | 1 mark for substituting into formula
1 mark for working that leads to required answer

$= \left| \dfrac{a - \frac{1}{2}(a - \frac{1}{a})}{1 + a \cdot \frac{1}{2}(a - \frac{1}{a})} \right|$ ✓

$= \left| \dfrac{a - \frac{1}{2}a + \frac{1}{2a}}{1 + \frac{a^2}{2} - \frac{1}{2}} \right|$

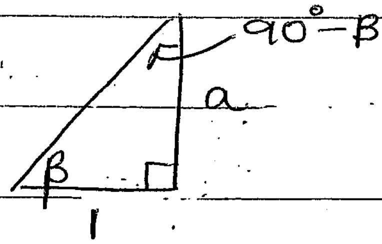
$= \left| \dfrac{\frac{a}{2} + \frac{1}{2a}}{\frac{1}{2} + \frac{a^2}{2}} \right|$

$\tan\theta = \left| \dfrac{a^2+1}{2a} \div \dfrac{1+a^2}{2} \right|$

$= \left| \dfrac{\cancel{2}}{\cancel{2}a} \right|$

$\tan\theta = \dfrac{1}{a}$ ✓

iv) $\tan\beta = \alpha$.



$\therefore \cot(90° - \beta) = \dfrac{1}{a} = \tan\theta$ ✓ ← 1 mark

$\therefore 90° - \beta = \theta$ ✓ ← 1 mark.

$\therefore \theta + \beta = 90°$

v) If $\theta = \beta$

$\theta + \beta = 2\beta = 90° \quad \therefore \beta = 45°$ ✓ ← 1 mark

$\tan 45° = a \quad \therefore a = 1$

$\therefore P$ is $(2, 1)$ ✓ ← 1 mark

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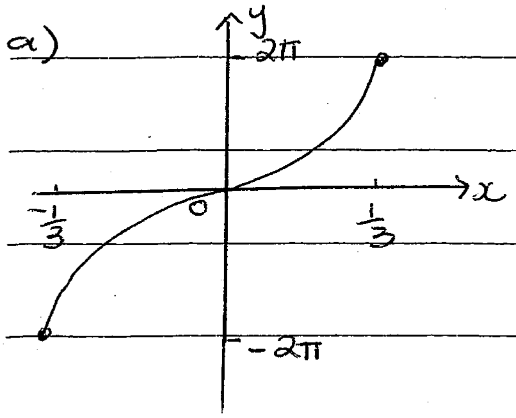
Name: Question 5 /10 Teacher:

a)



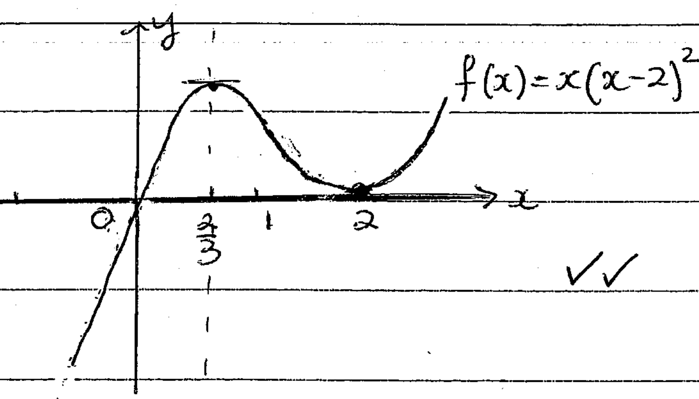
✓

$D: -\frac{1}{3} \le x \le \frac{1}{3}$ ✓

$R: -2\pi \le y \le 2\pi$

1 mark for graph
1 mark for both domain & range.

b) i)



✓✓

1 mark for shape
1 mark for showing y-intercept, where turning pts. are

ii) $a = \frac{2}{3}$ ✓. If $x \le \frac{2}{3}$, $f(x)$ is a 1:1 fn ∴ its inverse will exist. ✓

1 mark for 'a'.
1 mark for reason

iii) If $x = \frac{2}{3}$, $y = 1\frac{5}{27}$

$D_f: x \le \frac{2}{3}$ $R_f: y \le 1\frac{5}{27}$

$\therefore D_{f^{-1}}: x \le 1\frac{5}{27}$ ✓ ← 1 mark.

1

c) $\sin^{-1}x - \cos^{-1}y = \frac{\pi}{12}$ ①

$\sin^{-1}y + \cos^{-1}x = \frac{5\pi}{12}$ ②

$\underbrace{\sin^{-1}x + \cos^{-1}x} - \cos^{-1}y + \sin^{-1}y$ ① + ②

$\frac{\pi}{2} - \cos^{-1}y + \sin^{-1}y = \frac{\pi}{2}$ ✓ ← 1 mark

$\sin^{-1}y = \cos^{-1}y$

when $y = \frac{1}{\sqrt{2}}$ sub in ① ✓ 1 mark for 'y'

$\sin^{-1}x - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{12}$

$\sin^{-1}x = \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}$

$\therefore x = \frac{\sqrt{3}}{2}$ ✓ 1 mark for 'x'.

$\therefore$ the soln is $\left(\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}\right)$

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Name: Question 6 /10 Teacher: ______

a) $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= \sin^{-1}\frac{x}{2} + c$ ✓ ← 1 mark. Do not take mark off if no 'c'

b) $y = \cos^{-1}(3-2x)$ let $u = 3-2x$

$y' = dy/du \times du/dx$

$= \frac{-1}{\sqrt{1-(3-2x)^2}} \times -2$ ✓ ← 1 mark

$= \frac{2}{\sqrt{(1+3-2x)(1-(3-2x))}}$

$= \frac{2}{\sqrt{(4-2x)(2x-2)}}$ ✓ ← 1 mark for factorising denominator

$= \frac{2}{\sqrt{4(2-x)(x-1)}}$

$= \frac{1}{\sqrt{(2-x)(x-1)}}$ ✓ ← 1 mark for simplest factored form.

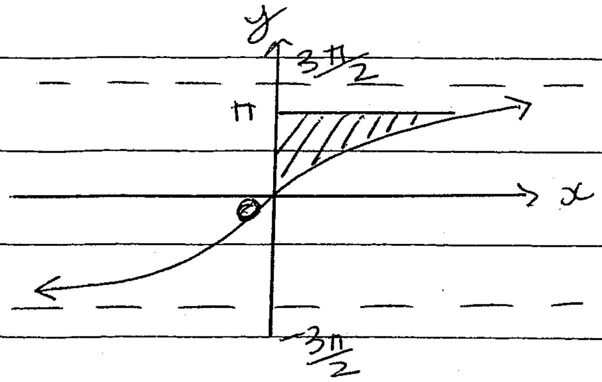
c) $\int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+9x^2}$

$= \frac{1}{9}\int_0^{1/\sqrt{3}} \frac{dx}{(\frac{1}{9}+x^2)}$

$= \frac{1}{9}\left[3\tan^{-1}3x\right]_0^{1/\sqrt{3}}$ ✓ ← 1 mark for primitive fn

$= \frac{1}{9}\left[3\tan^{-1}\frac{3}{\sqrt{3}} - 0\right]$

$= \frac{1}{9}\left[3\tan^{-1}\sqrt{3}\right] = \frac{1}{3}\times\frac{\pi}{3} = \frac{\pi}{9}$ ✓ ← 1 mark for value exact c)



$y = 3\tan^{-1}x$

$x = \tan\frac{y}{3}$ ✓ — 1 mark for making 'x' the subject

$\therefore V = \pi\int_0^{\pi} \tan^2\frac{y}{3}\,dy$

$= \pi\int_0^{\pi}\left(\sec^2\frac{y}{3} - 1\right)dy$ ✓ — 1 mark for writing in terms of $\sec^2\theta$.

$= \pi\left[3\tan\frac{y}{3} - y\right]_0^{\pi}$ ✓ — 1 mark for integrating correctly

$= \pi\left[3\tan\frac{\pi}{3} - \pi - 0\right]$

$= \pi\left[3\sqrt{3} - \pi\right]$ units^3 ✓ ← 1 mark for exact volume